



Invariants of Lie algebroids

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Abstract

Several new invariants of Lie algebroids have been discovered recently. We give an overview of these invariants and establish several relationships between them.

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1. Introduction

It is becoming increasingly apparent that Lie algebroids provide the appropriate setting for developing the differential geometry of singular geometric structures. The study of global properties of Lie algebroids is therefore a way of approaching the global theory of singular geometric structures, about which little is known. In this survey we shall describe several constructions of Lie algebroid invariants which have been introduced in the last few years. Although we are just starting to grasp their properties, it is clear that they play an important role in understanding the global behavior of singular geometric structures.

Let $\pi : A \rightarrow M$ be a Lie algebroid over M with anchor $\# : A \rightarrow TM$ and Lie bracket

$$[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A).$$

For general definitions and conventions we refer the reader to [3], and further background material is given in [24,25]. We shall adopt the point of view that such a Lie algebroid describes a concrete geometric situation. In each case, the Lie algebroid plays the role of the *tangent bundle*, and many constructions to be given later can be traced back to this simple idea. Some typical cases we have in mind are:

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Ordinary geometry. We take $A = TM$, $\#$ the identity, and $[\cdot, \cdot]$ the usual Lie bracket of vector fields. Later, we will see that virtually every construction for an arbitrary Lie algebroid reduces to some well-known construction when applied to this case.

Lie theory. At the other extreme, we take $A = \mathfrak{g}$ to be a Lie algebra and $M = \{*\}$ a single point set. Although this is a somewhat degenerate case, it is useful because the local-global dichotomy for Lie algebroids resembles Lie theory. Also, the terminology is usually motivated either by this case or the previous one.

Equivariant geometry. If we are given an action of a Lie algebra on a manifold, i.e., a homomorphism $\rho: \mathfrak{g} \rightarrow \mathcal{X}(M)$ to the Lie algebra of vector fields on M , then we have a naturally associated action Lie algebroid. The bundle A is the trivial vector bundle $M \times \mathfrak{g} \rightarrow M$, the anchor $\#: M \times \mathfrak{g} \rightarrow TM$ is defined by

$$\#(x, v) \equiv \rho(v)|_x,$$

and the Lie bracket is given by

$$[v, w](x) = [v(x), w(x)] + (\rho(v(x)) \cdot w)|_x - (\rho(w(x)) \cdot v)|_x,$$

where we identify a section v of $M \times \mathfrak{g} \rightarrow M$ with a \mathfrak{g} -valued function $v: M \rightarrow \mathfrak{g}$. In this case we have a particularly simple geometric interpretation for the orbit foliation of the algebroid,² even though the action does not always integrate to a global Lie group action.

Foliation theory. Let \mathcal{F} be a regular foliation of M . The associated involutive distribution $A = T\mathcal{F}$ has a Lie algebroid structure with anchor the inclusion into TM and bracket the Lie bracket of tangent vector fields to \mathcal{F} . Many constructions in Lie algebroid theory, related to the geometry and topology of the orbit foliation, are inspired by constructions in foliation theory.

Poisson geometry. Consider a Poisson manifold (M, π) , where $\pi \in \Gamma(\wedge^2 TM)$ is a bi-vector field satisfying $[\pi, \pi] = 0$. It is well known that the cotangent bundle $A = T^*M$ has a natural Lie algebroid structure, where the anchor $\#: T^*M \rightarrow TM$ is contraction with π , and the bracket on 1-forms is the Koszul bracket:

$$[\alpha, \beta] = \mathcal{L}_{\# \alpha} \beta - \mathcal{L}_{\# \beta} \alpha - d\pi(\alpha, \beta).$$

Many concepts to be discussed below were first introduced for Poisson manifolds and then generalized to Lie algebroids.

As a first example of an invariant let us consider Lie algebroid cohomology (see [24] for more details). We just mimic the usual definition of de Rham cohomology: the space of differential forms is $\Omega^r(A) = \Gamma(\wedge^r A^*)$, and we define the exterior differential

$$d_A: \Omega^\bullet(A) \rightarrow \Omega^{\bullet+1}(A)$$

² In this paper, foliations can be *singular* as in Sussmann [29]. By a *regular foliation* we mean a non-singular foliation.

by

$$\begin{aligned} d_A Q(\alpha_0, \dots, \alpha_r) &= \frac{1}{r+1} \sum_{k=0}^{r+1} (-1)^k \# \alpha_k (Q(\alpha_0, \dots, \hat{\alpha}_k, \dots, \alpha_r)) \\ &\quad + \frac{1}{r+1} \sum_{k < l} (-1)^{k+l+1} Q([\alpha_k, \alpha_l], \alpha_0, \dots, \hat{\alpha}_k, \dots, \hat{\alpha}_l, \dots, \alpha_r), \end{aligned} \quad (1.1)$$

where $\alpha_0, \dots, \alpha_r$ are any sections of A . In this way we obtain a complex $(\Omega^\bullet(A), d_A)$, and the corresponding cohomology is called the *Lie algebroid cohomology* of A (with trivial coefficients) and denoted $H^\bullet(A)$. The (dual of the) anchor induces a map

$$\#^*: H_{\text{de Rham}}^\bullet(M) \rightarrow H^\bullet(A),$$

which is usually neither injective nor surjective. For the geometric situations discussed above we obtain well-known cohomology theories such as de Rham cohomology, Lie algebra cohomology, foliated cohomology and Poisson cohomology. As these examples show, Lie algebroid cohomology may not be homotopy invariant and hence it may be hard to compute (to say the least).

The problem of computing Lie algebroid cohomology is intimately related with the singular behavior of the orbit foliation of the Lie algebroid. The same will be true about all other invariants to be introduced below, and this is in fact one of the main topics of the present work. The construction of the new invariants resembles the construction of Lie algebroid cohomology, in as much as, if one knows the proper conceptual general definitions, then the appropriate construction will be similar to the corresponding construction in standard geometry.

The plan of this paper is as follows. First we consider the fundamental group(oid) of a Lie algebroid (Section 2), which was introduced in [8] for the purpose of integrating Lie algebroids to Lie groupoids, and which is inspired by the construction of the fundamental group(oid) of a manifold. Then we consider non-linear and linear holonomy (Sections 3 and 4) in the spirit of foliation theory, which was defined in [14] for Lie algebroids. These will lead us naturally to primary and secondary characteristic classes (Section 5) for Lie algebroids, which were introduced in [6,14,22]. The last invariant we shall discuss is K -theory (Section 6) which was introduced in [16], and may be considered as an extension of ordinary topological K -theory.

2. The Weinstein groupoid

In [8], for any Lie algebroid A we have constructed a topological groupoid $\mathcal{G}(A)$, called the *Weinstein groupoid*, and which is a fundamental invariant of A . It should be thought of as the “monodromy groupoid” or “fundamental groupoid” of A : it is given as the set of equivalence classes of A -paths under A -homotopy

$$\mathcal{G}(A) = P(A) / \sim, \quad (2.1)$$

where:

- $P(A)$ denotes the set of A -paths, i.e., paths $a: I \rightarrow A$ on the interval $I = [0, 1]$, such that

$$\frac{d}{dt} \pi(a(t)) = \#a(t).$$

An A -path can be identified with a Lie algebroid morphism $TI \rightarrow A$. From a differential-geometric point of view, these are precisely the paths along which parallel transport can be performed whenever a connection has been chosen (see below and [14,15]).

- \sim denotes an equivalence relation, called *homotopy of A -paths*, that can be described at an abstract level as follows: two A -paths a_0 and a_1 are homotopic iff there exists a Lie algebroid homomorphism $T(I \times I) \rightarrow A$, which covers a (standard) homotopy between the base paths $\pi(a_i(t))$, with fixed end-points, and which restricts to $a_i(t)$ on the boundaries.

The groupoid structure on $\mathcal{G}(A)$ comes from concatenation of A -paths, which becomes associative when one passes to the quotient. The structure maps are the obvious ones (as in the monodromy groupoid).

To be able to work with \sim one needs more concrete descriptions which are furnished in [8]. There we show, for example, that this equivalence relation is the orbit equivalence relation of a Lie algebra action. Namely, the Lie algebra of time-dependent sections of A , vanishing at the end-points,

$$P_0\Gamma(A) = \{I \ni t \mapsto \eta_t \in \Gamma(A) : \eta_0 = \eta_1 = 0, \eta \text{ is of class } C^2 \text{ in } t\}$$

acts on $P(A)$. So we have a Lie algebra homomorphism

$$P_0\Gamma(A) \rightarrow \mathcal{X}(P(A)), \quad \eta \mapsto X_\eta$$

for which the image is precisely the tangent space to the orbits of \sim .

Clearly, one cannot expect the differentiable structure on the path space to go over to the quotient. One can show (see [4,8]) that this action gives a smooth foliation on the Banach manifold $P(A)$, for which the orbits are smooth submanifolds of finite codimension equal to $\dim M + \text{rk } A$. So the most one can say, in general, is that the Weinstein groupoid is of the same topological type as the orbit space of a foliation.

In order to give the precise obstructions for the Weinstein groupoid to be a Lie groupoid, and hence also the obstructions to integrating a Lie algebroid, we introduce certain *monodromy groups* of the Lie algebroid. For that purpose, observe that an element in the isotropy Lie algebra $\mathfrak{g}_x = \text{Ker}(\#_x)$ determines a constant A -path, and so we can set:

Definition 2.1. For each $x \in M$, the *monodromy group* based at x is the subgroup $N_x(A) \subset A_x$ consisting of those elements $v \in Z(\mathfrak{g}_x)$ which are homotopic to zero as A -paths.

Since the monodromy group $N_x(A)$ lies in the center $Z(\mathfrak{g}_x)$ of the isotropy Lie algebra, we can identify it with an abelian subgroup of the simply connected Lie group $\mathcal{G}(\mathfrak{g}_x)$ integrating the Lie algebra \mathfrak{g}_x . Henceforth, we use this identification with no further comment.

The obstructions to integrability are related to the lack of discreteness of the monodromy groups. To explain this, let us observe that the monodromy based at x arises as the image of a second order monodromy map $\partial : \pi_2(L, x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$ which relates the topology of the leaf L through x with the simply-connected Lie group $\mathcal{G}(\mathfrak{g}_x)$ integrating the isotropy Lie algebra $\mathfrak{g}_x = \text{Ker}(\#_x)$. From a conceptual point of view, the monodromy map can be viewed as an analogue of a boundary map of the homotopy long exact sequence of a fibration. Namely, if we consider the short exact sequence

$$0 \rightarrow \mathfrak{g}_L \rightarrow A_L \xrightarrow{\#} TL \rightarrow 0$$

as analogous to a fibration, the first few terms of the associated long exact sequence will be

$$\cdots \rightarrow \pi_2(L, x) \xrightarrow{\partial} \mathcal{G}(\mathfrak{g}_x) \rightarrow \mathcal{G}(A)_x \rightarrow \pi_1(L, x).$$

We have shown in [8] that $\text{Im } \partial$ lies in the center of $\mathcal{G}(\mathfrak{g}_x)$ and its intersection with the connected component of the identity of $Z(\mathcal{G}(\mathfrak{g}_x))$ coincides with $N_x(A)$. With these notations we have the following fundamental result:

Theorem 2.2 (Obstructions to Integrability [8]). *For a Lie algebroid A over M , the following are equivalent:*

- (i) A is integrable;
- (ii) The monodromy groups are uniformly discrete.

Let us be more precise about (ii). In order to measure the discreteness of the groups $N_x(A)$, set

$$r(x) = d(0, N_x(A) - \{0\}),$$

where the distance is computed with respect to an arbitrary norm on the vector bundle A and we adopt the convention $d(0, \emptyset) = +\infty$. Notice that $N_x(A) \subset A_x$ is discrete iff $r(x) > 0$. Then condition (ii) can be stated as

- (iia) For all $x \in M$, $r(x) > 0$;
- (iib) For all $x \in M$, $\liminf_{y \rightarrow x} r(y) > 0$.

Since the monodromy groups $N_x(A)$ are isomorphic as x varies in a leaf, (iia) is an obstruction along the leaves, while (iib) is an obstruction transverse to the leaves.

These obstructions are computable in many examples. Given any splitting $\sigma : TL \rightarrow A_L$ of the short exact sequence above, the *curvature* of σ is the \mathfrak{g}_L -valued 2-form $\Omega \in \Omega^2(L; \mathfrak{g}_L)$ defined by:

$$\Omega(X, Y) \equiv \sigma([X, Y]) - [\sigma(X), \sigma(Y)].$$

Assume that there exists a splitting such that this 2-form takes values in the center $Z(\mathfrak{g}_L)$. Then the monodromy map $\partial : \pi_2(L, x) \rightarrow \nu^*(L)$ is given by

$$\partial([\gamma]) = \int_{\gamma} \Omega, \tag{2.2}$$

and this gives an effective procedure to compute the monodromy in many example (see [8, Section 3.4]). Note that in this case, $Z(\mathfrak{g}_L)$ is canonically a flat vector bundle over L . The corresponding flat connection can be expressed with the help of the splitting σ as

$$\nabla_X \alpha = [\sigma(X), \alpha], \tag{2.3}$$

and it is easy to see that the definition does not depend on σ . In this way Ω_{σ} appears as a 2-cohomology class with coefficients in the local system defined by $Z(\mathfrak{g}_L)$ over L , and then the integration (2.2) is just the usual pairing between cohomology and homotopy. In practice one can always avoid working with local coefficients: if $Z(\mathfrak{g}_L)$ is not already trivial as a vector bundle, one can achieve this by pulling back to the universal cover of L (where parallel transport with respect to the flat connection gives the desired trivialization).

The connection given by (2.3) is the *Bott connection* on A_L , which can be introduced even for singular leaves. This connection will be discussed further below, when we consider the theory of holonomy for Lie algebroids, therefore providing a relation between these two invariants.

Many, if not all, results on integrability of Lie algebroids (see, e.g., [1,10,12,26,27,31]) follow from Theorem 2.2. We give a few examples and refer the reader to [8] for further examples and details.

Example 2.3. Let $A = TM$ be the tangent bundle Lie algebroid structure. A TM -path is just an ordinary path in M (given by its derivative), and a TM -homotopy is an ordinary homotopy in M with fixed end points, so we have

$$\mathcal{G}(TM) = \{(x, [\gamma], y) : x, y \in M, \gamma \text{ is a path from } x \text{ to } y\}.$$

Therefore, $\mathcal{G}(TM)$ is just the fundamental groupoid $\pi_1(M)$.

More generally, for any foliation \mathcal{F} of M , we have the fundamental groupoid $\pi_1(\mathcal{F})$, where now $T\mathcal{F}$ -paths are just \mathcal{F} -paths (paths lying on any fixed leaf) and $T\mathcal{F}$ -homotopies are homotopies within the set of \mathcal{F} -paths.

In both these cases the obstructions obviously vanish, and this corresponds to the well-known fact that $\pi_1(M)$ and $\pi_1(\mathcal{F})$ are Lie groupoids.

Example 2.4. Let \mathfrak{g} be a Lie algebra. Again, there are no obstructions and Theorem 2.2 gives Lie's third theorem. The construction of $\mathcal{G}(\mathfrak{g})$ given above coincides with the construction of the simply-connected Lie group integrating \mathfrak{g} which is given in the recent monograph of Duistermaat and Kolk [11].

Example 2.5. Let us give an example of a non-integrable Lie algebroid. Recall (see, e.g., [24]) that any closed two-form $\omega \in \Omega^2(M)$ defines a Lie algebroid structure on $A_\omega = TM \oplus \mathbb{L}$, where $\mathbb{L} = M \times \mathbb{R}$ is the trivial line bundle over M , the anchor is $(X, \lambda) \mapsto X$ and the Lie bracket is defined by

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f) + \omega(X, Y)).$$

Using the obvious splitting of A , which has curvature $\Omega_\sigma = \omega$, we see that the monodromy group based at x is given by

$$N_x(A_\omega) = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

and so coincides with the group of spherical periods of ω . If this group is non-discrete we obtain a non-integrable Lie algebroid.

For example, on the 2-sphere S^2 denote by ω_{S^2} the standard area form, and let $M = S^2 \times S^2$ with the closed 2-form $\omega_\lambda = \omega_{S^2} \oplus \lambda \omega_{S^2}$, where $\lambda \in \mathbb{R}$. Then the monodromy group $N_x(A_{\omega_\lambda})$ is discrete iff λ is rational.

The reader will notice that in the symplectic case this obstruction is Kostant's "prequantization condition".

Example 2.6. Let us consider the case of a Poisson manifold. In Poisson geometry, A -paths are also called *cotangent paths* (see [14,17]). Cotangent homotopies are given by an action of the Lie algebra $P_0\Omega(M)$ of time-dependent 1-forms, vanishing at the end-points, on the space of cotangent paths $P(T^*M)$. The orbits of this action have codimension $2 \dim M$. In this case, this action is the restriction of a Lie algebra

action on the larger space of *all* paths $\tilde{P}(T^*M) = \{a : I \rightarrow T^*M\}$, which is tangent to the submanifold $P(T^*M) \subset \tilde{P}(T^*M)$.

Since we have a natural identification $\tilde{P}(T^*M) \simeq T^*P(M)$, where $P(M)$ denotes the Banach space of paths $\gamma : I \rightarrow M$ which are piecewise of class C^2 ,³ we have a natural symplectic structure on $\tilde{P}(T^*M)$. This brings symplectic geometry into the picture, and we have the following result which is proved in [4]:

Proposition 2.7. *The Lie algebra action of $P_0\Omega(M)$ on $\tilde{P}(T^*M)$ is Hamiltonian with equivariant moment map $J : \tilde{P}(T^*M) \rightarrow P_0\Omega(M)^*$ given by*

$$\langle J(a), \eta \rangle = \int_0^1 \left\langle \frac{d}{dt} \pi(a(t)) - \#a(t), \eta(t, \gamma(t)) \right\rangle dt.$$

Since the level set $J^{-1}(0)$ is precisely the set of cotangent paths $P(T^*M)$, it follows that in this case the Weinstein groupoid can be described alternatively as a Marsden–Weinstein reduction:

$$\mathcal{G}(T^*M) = \tilde{P}(T^*M) / P_0\Omega(M). \quad (2.4)$$

The two alternative descriptions (2.1) and (2.4) give the precise relationship between the integrability approach introduced in [8], which as we have explained is valid for any Lie algebroid, and the approach of Cattaneo and Felder in [4], which is based on the Poisson sigma-model and which only holds for Poisson manifolds.

Since we have the alternative description of the Weinstein groupoid of a Poisson manifold as a Marsden–Weinstein reduction we obtain a symplectic form on $\mathcal{G}(T^*M)$. If this groupoid is smooth this symplectic form is compatible with the groupoid structure, and we conclude that

Theorem 2.8 (Symplectic integration [4,9]). *If the Weinstein groupoid $\mathcal{G}(T^*M)$ of a Poisson manifold M is smooth, then it is a symplectic groupoid integrating M .*

Obviously, it is possible to form quotients of the groupoid $\mathcal{G}(T^*M)$ outside the symplectic category, to yield examples of groupoids integrating (M, π) and which are not symplectic integrations. Results on the integrability of regular Poisson manifolds are given in [1,10], and for a detailed discussion we refer the reader to the forthcoming article [9].

3. Holonomy for Lie algebroids

The theory of holonomy for algebroids relies, as in the case of foliations, on the concept of *connection*, since one wants to compare the transverse Lie algebroid structure as we vary along a leaf. Let us start by recalling the general notion of connection which we have introduced in [14].

³ We need cotangent paths to be piecewise of class C^1 , so we require their base paths to be piecewise of class C^2 . See also [8].

Definition 3.1. Let $p : E \rightarrow M$ be a fiber bundle over M , and $\pi : A \rightarrow M$ a Lie algebroid over M . An A -connection on E is a bundle map $h : p^*A \rightarrow TE$ which makes the following diagram commute:

$$\begin{array}{ccc} p^*A & \xrightarrow{h} & TE \\ \hat{p} \downarrow & & \downarrow p_* \\ A & \xrightarrow{\#} & TM \end{array}$$

Note that in this definition $h : p^*A \rightarrow TE$ is a generalization of the notion of *horizontal lift* of tangent vectors that one finds in the usual theory of connections: For any $a \in A$, $h(u, a)$ is the horizontal lift of a to the point $u \in E$ in the fiber over $x = \pi(a)$, and the diagram means that $p_*h(u, a) = \#a$. Instead of lifting tangent vectors in TM we lift elements of A , the bundle that replaces the tangent bundle.

Given some A -path $a : I \rightarrow A$, and a point $u_0 \in E$ in the fiber over the initial base point $x_0 = \pi(a(0))$, we can look for the horizontal lift $\gamma : I \rightarrow E$, i.e., the unique curve satisfying:

$$\dot{\gamma}(t) = h(\gamma(t), a(t)), \quad \gamma(0) = u_0.$$

Note that this system has a solution defined only for small $t \in [0, \varepsilon)$. However, suppose that E is a vector bundle and that the connection has the property that $h(u, 0) = 0$. Then, it follows from standard results in the theory of o.d.e.'s, that the solution will be defined for all $t \in [0, 1]$, provided we choose the initial condition u_0 small enough. In this way, we get a diffeomorphism from a neighborhood of zero in the fiber E_{x_0} over the initial point onto a neighborhood of zero in the fiber E_{x_1} over the final point. Such a map is called, of course, *parallel transport* along the A -path $a : I \rightarrow A$.

Let us consider now a fixed leaf $i : L \hookrightarrow M$ of the Lie algebroid $\pi : A \rightarrow M$. We denote by $\nu(L) = T_L M / TL$ the normal bundle to L and by $p : \nu(L) \rightarrow L$ the natural projection. By the tubular neighborhood theorem, there exists a smooth immersion $\tilde{i} : \nu(L) \rightarrow M$ satisfying the following properties:

- (i) $\tilde{i}|_Z = i$, where we identify the zero section Z of $\nu(L)$ with L ;
- (ii) \tilde{i} maps the fibers of $\nu(L)$ transversely to the foliation of M ;

We shall define an A_L -connection on the normal bundle $\nu(L)$, so that parallel transport for this connection will be the Lie algebroid holonomy.

Assume that we have fixed such an immersion, and let $x \in L$. Each fiber $F_x = p^{-1}(x)$ is a submanifold of M transverse to the foliation, and so we have the transverse Lie algebroid structure $A_{F_x} \rightarrow F_x$ (see [14]). Because F_x is a linear space we can choose a trivialization and identify the fibers $(A_{F_x})_u$ for different $u \in p^{-1}(x)$. Finally, we choose a complementary vector subbundle $E \subset A$ to A_{F_x} :

$$A_u = E_u \oplus (A_{F_x})_u. \quad (3.1)$$

Note that, by construction, the anchor $\# : A \rightarrow TM$ maps A_{F_x} onto TF_x , its restriction to E is injective, and vectors in $\#E$ are tangent to the orbit foliation.

Let $\alpha \in A_x$. We decompose α according to (3.1):

$$\alpha = \alpha^\parallel + \alpha^\perp, \quad \text{where } \alpha^\parallel \in E_x, \alpha^\perp \in (A_{F_x})_x.$$

For each $u \in F_x = p^{-1}(x)$, we denote by $\tilde{\alpha}_u^\parallel \in E_u$ the unique element such that $d_u p \cdot \# \tilde{\alpha}_u^\parallel = \# \alpha^\parallel$, and by $\tilde{\alpha}_u^\perp \in (A_{F_x})_u$ the element corresponding to α^\perp under the identification $(A_{F_x})_u \simeq (A_{F_x})_x$. We also set $\tilde{\alpha} \equiv \tilde{\alpha}^\parallel + \tilde{\alpha}^\perp$.

We can now define our connection: given $\alpha \in A_x$, $x \in L$, and $u \in F_x$, the horizontal lift to $\nu(L)$ is the map

$$h(u, \alpha) = \# \tilde{\alpha}_u \in T_u \nu(L).$$

By construction, we have the defining property of an A -connection:

$$p_* h(u, \alpha) = \# \alpha, \quad u \in p^{-1}(x).$$

The definition of h depends on several choices made: tubular neighborhood, trivialization of A_{F_x} and complementary vector bundle E . The changes of choices will eventually lead to conjugate holonomy homomorphisms (to be defined below).

Given an A -path $a: I \rightarrow A$ with initial point x_0 and final point x_1 , parallel transport gives us a diffeomorphism $H_L(a)_0: F_{x_0} \rightarrow F_{x_1}$, defined on neighborhoods of the origin. More is true: $H_L(a)_0$ is covered by a Lie algebroid isomorphisms $H_L(a)$ from $A_{F_{x_0}}$ to $A_{F_{x_1}}$, so we have

$$\begin{array}{ccc} A_{F_{x_0}} & \xrightarrow{H_L(a)} & A_{F_{x_1}} \\ \downarrow & & \downarrow \\ F_{x_0} & \xrightarrow{H_L(a)_0} & F_{x_1} \end{array}$$

This can be seen as follows. Given any A -path $a(t)$ in L , we can find a time-dependent section α_t of A over L such that $\alpha_t(\gamma(t)) = a(t)$, where $\gamma(t) = \pi(a(t))$. Now we can define a time-dependent section $\tilde{\alpha}_t$ covering α_t such that the horizontal lift of $a(t)$ is an integral curve of the time-dependent vector field

$$X_t = \# \tilde{\alpha}_t,$$

so that $H_L(a)_0$ is the map induced by the time-1 flow of X_t on F_{x_0} . Since the flow of X_t is induced by the 1-parameter family of Lie algebroid homomorphisms $\Phi_t^{\alpha_t}$ of A obtained by integrating the family $\tilde{\alpha}_t$, the homomorphisms $\Phi_1^{\alpha_t}$ give a Lie algebroid isomorphism $H_L(a)$ from $A_{F_{x_0}}$ to $A_{F_{x_1}}$, which covers $H_L(a)_0$.

Since $H_L(a)$ is the time-1 map of some flow it follows that if a' is another A -path in L such that $x_1 = x'_0$ we have

$$H_L(a \cdot a') = H_L(a) \circ H_L(a'), \quad (3.2)$$

where the dot denotes concatenation of A -paths. We call $H_L(a)$ the A -holonomy of the A -path $a(t)$. One extends the definition of H_L for piecewise smooth A -paths in the obvious way.

Denote by $\mathfrak{Aut}(A_{F_x})$ the group of germs at 0 of Lie algebroid automorphisms of A_{F_x} which map 0 to 0, and by $\Omega_A(L, x_0)$ the group of piecewise smooth A -loops based at x_0 .

Definition 3.2. The A -holonomy of the leaf L with base point x_0 is the map

$$H_L: \Omega_A(L, x_0) \rightarrow \mathfrak{Aut}(A_{F_{x_0}}).$$

Notice that the holonomy of a leaf L depends on the tubular neighborhood $\tilde{i}: \nu(L) \rightarrow M$, on the choice of trivialization, and on the choice of complementary bundles. However, two different choices

lead to conjugate homomorphisms. This Lie algebroid holonomy has, however, a major drawback: two A -paths whose base paths are homotopic may have distinct holonomy. On the other hand, one can show, that if they are homotopic as A -paths they lead to the same holonomy, so we do get a homomorphism

$$H_L : \mathcal{G}(A)_x^x \rightarrow \mathfrak{Aut}(A_{F_x}).$$

For practical computations it is much more efficient to have a homomorphism defined on the fundamental group $\pi_1(L, x)$. In [15], following constructions given in [15] and [17] for the Poisson case, we have introduced a notion of *reduced holonomy* which is homotopy invariant relative to the base paths. Recall that $\mathfrak{Aut}(A_{F_x})$ denotes the group of germs at 0 of Lie algebroid automorphisms of A_{F_x} which map 0 to 0. By an *inner Lie algebroid automorphism* of A we mean an automorphism which is the time-1 flow of some time-dependent section. We shall denote by $\mathfrak{Out}(A_{F_x})$ the corresponding group of germs of *outer Lie algebroid automorphisms*.⁴

In [15] we prove the following

Proposition 3.3. *Let $x \in L \subset M$ be a leaf of A with associated A -holonomy $H_L : \Omega_A(L, x) \rightarrow \mathfrak{Aut}(F_x)$. If $a_1(t)$ and $a_2(t)$ are A -loops based at x with base paths $\gamma_1 \sim \gamma_2$ homotopic then $H_L(a_1)$ and $H_L(a_2)$ represent the same equivalence class in $\mathfrak{Out}(F_x)$.*

Given a loop γ in a leaf L we shall denote by $\overline{H}_L(\gamma) \in \mathfrak{Out}(A_{F_x})$ the equivalence class of $H_L(a)$ for some piecewise smooth family $a(t)$ with $\#a(t) = \gamma(t)$. The map

$$\overline{H}_L : \Omega(L, x) \rightarrow \mathfrak{Out}(A_{F_x})$$

will be called the *reduced holonomy homomorphism* of L . This map extends to continuous loops and, by a standard argument, it induces a group homomorphism $\overline{H}_L : \pi_1(L, x) \rightarrow \mathfrak{Out}(A_{F_x})$.

Recall that, for a foliation \mathcal{F} of a manifold M , a *saturated set* is a set $S \subset M$ which is a union of leaves of \mathcal{F} . A leaf L is called *stable* if it has arbitrarily small saturated neighborhoods. In the case of the orbit foliation of a Lie algebroid a set is saturated iff it is invariant under all inner automorphisms. Hence, a leaf is stable iff it has arbitrarily small neighborhoods which are invariant under all inner automorphisms.

We shall call a leaf L *transversely stable* if $N \cap L$ is a stable leaf for the transverse Lie algebroid structure A_N , i.e., if there are arbitrarily small neighborhoods of $N \cap L$ in N which are invariant under all inner automorphisms of A_N . Using this notion of Lie algebroid holonomy one can prove a Reeb-type stability theorem:

Theorem 3.4 (Stability Theorem [15]). *Let L be a compact, transversely stable leaf, with finite reduced holonomy. Then L is stable, i.e., L has arbitrarily small neighborhoods which are invariant under all inner automorphisms. Moreover, each leaf near L is a bundle over L whose fiber is a finite union of leaves of the transverse Lie algebroid structure.*

The local splitting theorem for Lie algebroids (see [15, Theorem 1.1]) states that locally a Lie algebroid splits as a product of A_L and A_N . Using holonomy one can investigate if a neighborhood of a leaf trivializes as in the local splitting theorem. An obvious necessary condition is that the Lie algebroid holonomy be trivial. We refer the reader to [14,15] for further details.

⁴ As usual, for a Lie algebroid A , the group of outer Lie algebroid automorphisms is the quotient $\text{Aut}(A)/\text{Inn}(A)$.

4. Linear holonomy

Recall (see [15,33]) that a *linear Lie algebroid* is a Lie algebroid $\pi : A \rightarrow V$ such that:

- (i) The base is a vector space V (so $\pi : A \rightarrow V$ is trivial);
- (ii) For this trivialization, the bracket of constant sections is a constant section;
- (iii) For this trivialization, the anchor of a constant section is a linear vector field.

The reader should notice that a linear Lie algebroid A is isomorphic to a (linear) action Lie algebroid $\mathfrak{g} \times V$.

For a Lie algebroid A , the transverse Lie algebroid structure to a leaf of A is a germ of a Lie algebroid for which the anchor vanishes at the base point. One should expect then that there is a well-defined linear approximation. In fact, we have the following proposition:

Proposition 4.1. *Let $A \rightarrow M$ be any Lie algebroid, fix $x_0 \in M$ and denote by L the leaf through x_0 . There is a natural linear Lie algebroid structure A^{lin} over the normal space $N_{x_0} = T_{x_0}M / T_{x_0}L$ with*

$$A^{\text{lin}} = \mathfrak{g}_{x_0} \times N_{x_0}.$$

We call $A^{\text{lin}} \rightarrow N_{x_0}$ the *linear approximation* to A at x_0 . We also have a linear version of holonomy

$$H_L^{\text{lin}} \equiv dH_L : \Omega_A(L, x) \rightarrow \text{Aut}(\mathfrak{g}) \times \text{GL}(F_x) \simeq \text{Aut}(A^{\text{lin}}).$$

Moreover, we can obtain linear holonomy as parallel transport along a linear A -connection generalizing the Bott connection of ordinary foliation theory, and which we will now recall.

First, if $p : E \rightarrow M$ is a vector bundle over M , a linear A -connection on E is a connection $h : p^*A \rightarrow TE$ which is linear:

$$h(u, \alpha_1 + \alpha_2) = h(u, \alpha_1) + h(u, \alpha_2), \quad h(u, \lambda\alpha) = \lambda h(u, \alpha),$$

where $\lambda \in \mathbb{R}$, $u \in E$ and $\alpha_i \in A$.

In exactly the same way as one does with ordinary connections, we can associate with a linear connection an operator $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ which satisfies

- (i) $\nabla_{\alpha_1 + \alpha_2} s = \nabla_{\alpha_1} s + \nabla_{\alpha_2} s$;
- (ii) $\nabla_{\alpha}(s_1 + s_2) = \nabla_{\alpha} s_1 + \nabla_{\alpha} s_2$;
- (iii) $\nabla_{f\alpha} s = f \nabla_{\alpha} s$;
- (iv) $\nabla_{\alpha}(fs) = f \nabla_{\alpha} s + \#_{\alpha}(f)s$,

where $\alpha, \alpha_1, \alpha_2 \in \Gamma(A)$, $s, s_1, s_2 \in \Gamma(E)$, and $f \in C^{\infty}(M)$. Conversely, any operator satisfying (i)–(iv) determines a linear connection. For a detailed discussion of such connections see [15].

The *Bott connection* of a Lie algebroid is a pair of linear A -connections on $\text{Ker}\#|_L$ and on $\nu^*(L)$. On one hand, we have on the vector bundle $\text{Ker}\#|_L$ the linear A -connection:

$$\nabla_{\alpha}^L \gamma \equiv [\tilde{\alpha}, \tilde{\gamma}], \tag{4.1}$$

where $\tilde{\alpha}, \tilde{\gamma} \in \Gamma(A)$ are sections extending the sections $\alpha \in \Gamma(A_L)$ and $\gamma \in \Gamma(\text{Ker}\#|_L)$. On the other hand, we have on the conormal bundle $\nu^*(L) = \{\omega \in T_L^*M: \omega|_{TL} = 0\}$ the linear A -connection:

$$\check{\nabla}_{\alpha}^L \omega \equiv \mathcal{L}_{\# \tilde{\alpha}} \tilde{\omega}|_x, \quad (4.2)$$

where now we take a section $\tilde{\alpha} \in \Gamma(A)$ and a 1-form $\tilde{\omega} \in \Omega^1(M)$ extending the sections $\alpha \in \Gamma(A_L)$ and $\omega \in \Gamma(\nu^*(L))$. It is easy to check that ∇^L and $\check{\nabla}^L$ satisfy properties (i)–(iv) above.

It is more convenient to consider the connections ∇^L and $\check{\nabla}^L$ together, rather than leaf by leaf. Also, we can work on the direct sum $A \oplus TM$ (the reason for the strange symbol will be explained later) and replace the pair of connections by a single connection, so we set:

Definition 4.2. A linear connection $\nabla = \nabla^A \oplus \nabla^M$ on $A \oplus TM$ is called a *basic connection* if

(i) ∇ is compatible with the Lie algebroid structure, i.e.,

$$\nabla^M \# = \# \nabla^A;$$

(ii) ∇^A and ∇^M restrict to the Bott connection on each leaf L , i.e., if $\alpha, \gamma \in \Gamma(A)$, $\omega \in \Omega^1(M)$, with $\# \gamma|_L = 0$ and $\omega|_{TL} = 0$, then

$$\nabla_{\alpha}(\gamma, \omega)|_L = ([\alpha, \gamma], \mathcal{L}_{\# \alpha} \omega)|_L.$$

Basic connections always exist. A simple procedure, due to Crainic, for constructing a basic connection is to start with any TM -connection $\tilde{\nabla}$ on A and set:

$$\nabla_{\alpha}(\gamma, X) = (\tilde{\nabla}_{\# \gamma} \alpha + [\alpha, \gamma], \# \tilde{\nabla}_X \alpha + [\# \alpha, X]).$$

The holonomy along a leaf L of a basic connection ∇ gives the linear holonomy of L introduced above in the following way: the holonomy of the basic connection ∇ determines endomorphisms of the fiber A_x which map $\ker \#_x$ isomorphically into itself, and these are the linear holonomy maps. Moreover, we have the following Bott-type vanishing theorem which lies at the basis of the secondary characteristic classes to be introduced in the next section.

Theorem 4.3 (Bott vanishing theorem [14]). *Let R denote the curvature of a basic connection. Then for any section $\gamma \in \Gamma(A)$ and 1-form $\omega \in \Omega^1(M)$ satisfying $\# \gamma|_L = 0$ and $\omega|_{TL} = 0$, we have*

$$R(\alpha, \gamma)(\gamma, \omega)|_L = 0.$$

For more on Bott vanishing theorems in the special context of regular Lie algebroids we refer to the work of Kubarski [22].

5. Characteristic classes

One can define (primary) characteristic classes for algebroids as one does in the usual Chern–Weil theory. For example, let $E \rightarrow M$ be a real vector bundle with rank q and pick some A -connection ∇ on E . Its curvature

$$R(\alpha, \beta) = \nabla_{\alpha} \nabla_{\beta} - \nabla_{\beta} \nabla_{\alpha} + \nabla_{[\alpha, \beta]},$$

defines a linear map $R_{\alpha,\beta} = R(\alpha, \beta): E_x \rightarrow E_x$ which satisfies $R_{\alpha,\beta} = -R_{\beta,\alpha}$. Hence, the map $(\alpha, \beta) \mapsto R_{\alpha,\beta}$ can be considered as a $\mathfrak{gl}(E)$ -valued 2-section, and by fixing a basis of local sections for E , so that $E_x \simeq \mathbb{R}^q$, we have that $R_{\alpha,\beta} \in \mathfrak{gl}_q(\mathbb{R})$. This matrix representation of $R_{\alpha,\beta}$ is defined only up to a change of basis in \mathbb{R}^q . Therefore, if

$$P: \mathfrak{gl}_q(\mathbb{R}) \times \cdots \times \mathfrak{gl}_q(\mathbb{R}) \rightarrow \mathbb{R}$$

is a symmetric, k -multilinear, $\text{Ad}(GL_q(\mathbb{R}))$ -invariant function, we can introduce a well-defined $2k$ -form $\lambda(\nabla)(P) \in \Omega^{2k}(A)$ by the formula

$$\lambda(\nabla)(P)(\alpha_1, \dots, \alpha_{2k}) = \sum_{\sigma \in S_{2k}} (-1)^\sigma P(R_{\alpha_{\sigma(1)}, \sigma(2)}, \dots, R_{\alpha_{\sigma(2k-1)}, \sigma(2k)}). \quad (5.1)$$

This form is closed and hence defines a certain Lie algebroid cohomology class $[\lambda(\nabla)(P)] \in H^{2k}(A)$. It is not hard to see that this cohomology class is independent of the choice of connection, so we have defined some intrinsic characteristic classes of the vector bundle E . For example, if we let P_k be the elementary symmetric polynomials we have the A -Pontrjagin classes

$$p_k(E, A) = [\lambda(P_{2k})] \in H^{4k}(A).$$

As usual, one does not need to consider the classes for odd k since we have

$$[\lambda(P_{2k-1})] = 0,$$

as can be seen by choosing a connection compatible with a Riemannian metric.

However, these classes do not really contain any new information. In fact, the anchor $\#: A \rightarrow TM$ determines a chain map $\#^*: (\Omega^\bullet(M), d) \rightarrow (\Omega^\bullet(A), d_A)$ and so we have an induced map in cohomology:

$$\#^*: H^\bullet(M) \rightarrow H^\bullet(A).$$

If we choose some ordinary connection $\tilde{\nabla}$ on E and take $\nabla_\alpha = \tilde{\nabla}_{\# \alpha}$ we see immediately that

$$p_k(E, A) = \#^* p_k(E),$$

where $p_k(E)$ are the usual Pontrjagin classes of E . The same is true for any (primary) characteristic classes one may define: we have a commutative diagram

$$\begin{array}{ccc} I^\bullet(G) & \longrightarrow & H_{\text{de Rham}}^\bullet(M) \\ & \searrow & \downarrow \#^* \\ & & H^\bullet(A) \end{array}$$

where on the top row we have the usual Chern–Weil homomorphism and on the diagonal we have the A -Chern–Weil homomorphism (see [14]). These classes were introduced for Poisson manifolds by Vaisman in [30], and for regular Lie algebroids by Kubarski in [23].

The fact that all these classes arise as image by $\#^*$ of well-known classes is perhaps a bit disappointing. However, one can define *secondary characteristic classes* which are true invariants of the Lie algebroid, in the sense that they do not arise as images by $\#^*$ of some de Rham cohomology classes. These classes are analogous to the exotic classes of foliation theory introduced by Bott et al. (see, e.g., [2]). To define them introduce a pair of connections (∇^1, ∇^0) on $A \oplus TM$ where:

- ∇^1 is a basic connection;
- ∇^0 is a Riemannian connection (i.e., $\nabla_\alpha^0 = \nabla_{\# \alpha}$ with ∇ the Levi-Civita connection).

Given an Ad-invariant, symmetric polynomial P , the classes are defined by a transgression formula in the spirit of Chern and Simons [5]:

$$\lambda^{1,0}(P)(\alpha_1, \dots, \alpha_{2k-1}) = k \sum_{\sigma \in S_{2k-1}} (-1)^\sigma \int_0^1 P(\nabla_{\alpha_{\sigma(1)}}^{1,0}, R_{\alpha_{\sigma(2)}, \alpha_{\sigma(3)}}^t, \dots, R_{\alpha_{\sigma(2k-2)}, \alpha_{\sigma(2k-1)}}^t) dt, \quad (5.2)$$

where $\nabla^{1,0} = \nabla^1 - \nabla^0$ and R^t is the curvature of $\nabla^t = (1-t)\nabla^0 - t\nabla^1$. Again we have:

Theorem 5.1 (Secondary classes [14]). *Let k be odd. Then*

- (i) $\lambda^{1,0}(P) \in \Omega^{2k-1}(A)$ is closed;
- (ii) The cohomology class $[\lambda^{1,0}(P)] \in H^{2k-1}(A)$ is independent of the choice of connections.

In general these classes do not lie in the image of $\#^*: H^\bullet(M) \rightarrow H^\bullet(A)$, as can be seen from some of the examples given below, so we obtain genuine invariants of the Lie algebroid. In particular, if we take $P = P_k$ the elementary symmetric polynomials we obtain the *secondary characteristic classes* of a Lie algebroid:

$$m_k(A) = [\lambda^{1,0}(P_k)] \in H^{2k-1}(A), \quad k = 1, 3, 5, \dots$$

Explicit computation of these classes appear in [14]. The best understood class is m_1 and it was known before [14] as the *modular class* of a Lie algebroid. This class was introduced first by Weinstein in [32], for the special case of Poisson manifolds, and by Weinstein et al. in [13] for general Lie algebroids. There the following geometric interpretation was given. Let us think of sections of the line bundle $Q_A = \wedge^{\text{top}} A \otimes \wedge^{\text{top}} TM$ (or $Q_A \otimes Q_A$ in the non-orientable case) as “transverse measures” in A . Then the modular class is the obstruction to the existence of invariant transverse measures: $m_1(A) = 0$ iff there exists a measure invariant under the flow of any section of A . In the Poisson case, $m_1(A) = 0$ iff there exists a (true) measure invariant under the flow of any hamiltonian diffeomorphism. The modular class was also studied in a purely algebraic context by Huebschmann [20,21] and Xu [34].

Crainic in [6] has developed, independently from [14], a theory of characteristic classes of representations. Recall that a *representation* of a Lie algebroid $A \rightarrow M$ is a vector bundle $E \rightarrow M$ together with a map $\Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ satisfying:

- (i) $(f\alpha) \cdot s = f\alpha \cdot s$;
- (ii) $\alpha \cdot (fs) = f\alpha \cdot s + \# \alpha(f)s$;
- (iii) $[\alpha, \beta] \cdot s = \alpha \cdot (\beta \cdot s) - \beta \cdot (\alpha \cdot s)$.

If we set $\nabla_\alpha s \equiv \alpha \cdot s$ then we see that a representation is nothing other than a flat A -connection on E . The terminology is motivated by the case when A is a Lie algebra.

Assume then that $E \rightarrow M$ is a representation of a Lie algebroid A and denote by ∇^1 the associated flat A -connection on E . Then ∇^1 induces an adjoint connection ∇^{1*} on the dual bundle E^* . If we pick some Riemannian metric on E we obtain an identification $E \simeq E^*$, so that the adjoint connection

determines a new A -connection ∇^0 on E . We can then check that if P is some Ad -invariant polynomial, the transgression formula (5.2) gives characteristic classes of the representation:⁵

$$u(E, P) = [\lambda_E^{1,0}(P)] \in H^\bullet(A).$$

These are the characteristic classes introduced by Crainic in [6]. Their main properties are given in the following proposition:

Proposition 5.2. *For representations E and F of a Lie algebroid A , and any characteristic class $u(\cdot) = u(\cdot, P)$ as defined above, we have:*

- (i) $u(E \oplus F) = u(E) + u(F)$;
- (ii) $u(E \otimes F) = \text{rk}(E)u(F) + \text{rk}(F)u(E)$;
- (iii) $u(E^*) = -u(E)$.

In particular, the characteristic classes vanish if E admits an invariant metric, hence these classes measure the obstruction to the existence of such invariant metrics.

Notice that the Crainic classes are representation dependent, while the classes introduced above were intrinsic classes of the Lie algebroid. If there existed some natural or canonical representation of a Lie algebroid A , one would expect that the (extrinsic) classes of such representation would give the intrinsic characteristic classes of A . However, a general Lie algebroid has no adjoint action as opposed to, say, a Lie algebra. It turns out that it is still possible to recover the intrinsic characteristic classes from characteristic classes of a representation if one weakens the notion of representation to a *representation up to homotopy*. Then there is a natural *adjoint representation up to homotopy* for every Lie algebroid A , and the Crainic classes of this representation yield the intrinsic characteristic classes.

The notion of connection up to homotopy is obtained as a weaker version of Quillen's (see [28]) notion of superconnection. Recall that a super-vector bundle is just a \mathbb{Z}_2 -graded vector bundle over a manifold M . If (E, ∂) is a super-complex of vector bundles over M :

$$E^1 \begin{smallmatrix} \xleftarrow{\partial_0} \\ \xrightarrow{\partial_1} \end{smallmatrix} E^0$$

we can view it as an element in the K -theory of M , i.e., the formal differences $E = E^0 \ominus E^1$. We set:

Definition 5.3. An A -connection up to homotopy on a super-vector bundle (E, ∂) is a \mathbb{R} -bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$, such that:

- (i) ∇ preserves the grading and $\nabla \partial_i = \partial_{i+1} \nabla$;
- (ii) $\nabla_\alpha(fs) = f \nabla_\alpha s + \# \alpha(f)s$;
- (iii) $\nabla_{f\alpha}s = f \nabla_\alpha s + [H(f, \alpha), \partial]s$,

where the homotopy $H(f, \alpha) : E \rightarrow E$ has degree 1.

⁵ The classes are actually defined only for complex representations and for polynomials representing classes in the relative cohomology $H^\bullet(GL(n), U(n))$, but we shall ignore this aspect here.

A flat connection up to homotopy is the same as a *representation up to homotopy* (see [13], for details and background on these connections). For representations up to homotopy one can define extrinsic characteristic classes $[\lambda_E^{1,0}(P)] \in H^\bullet(A)$ in a similar manner to the case of ordinary representations (see [7]). The main example we are interested in is the *adjoint representation up to homotopy* of a Lie algebroid A . It is the representation

$$\text{ad}(A): A \xrightleftharpoons[\#]{0} TM$$

where we take

$$\nabla_\alpha^{\text{ad}(A)}(\gamma, X) = ([\alpha, \gamma], \mathcal{L}_{\# \alpha} X).$$

The reader may check that this is indeed a flat connection up to homotopy with homotopy map $H(f, \alpha)$ given by:

$$H(f, \alpha)\gamma = 0, \quad H(f, \alpha)X = X(f)\alpha.$$

At this point the reader will notice the close relationship between the adjoint representations up to homotopy and the basic connections introduced in the previous section: every basic connection determines a flat connection up to homotopy on the super-vector bundle $A \oplus TM$. To make this relationship more precise, and finally justify the use of the symbol \oplus as a “difference” similar to K -theory, we introduce a notion of equivalence among connections up to homotopy:

Definition 5.4. Two connections up to homotopy ∇ and ∇' on a super-vector bundle E are said to be *equivalent* if there exists some degree-zero $\text{End}(E)$ -valued 1-form θ such that:

$$\nabla'_\alpha = \nabla_\alpha + [\theta(\alpha), \partial].$$

One can then show (see [7] for details):

Proposition 5.5. A true A -connection ∇ on $A \oplus TM$ is equivalent to $\nabla^{\text{ad}(A)}$ iff ∇ is a basic connection.

Finally, Crainic has established that the intrinsic characteristic classes of a Lie algebroid coincide with the characteristic classes of the adjoint representation:

$$[\lambda_{\text{ad}(A)}^{1,0}(P)] = [\lambda^{1,0}(P)].$$

Namely, he shows that, in general, two equivalent representations up to homotopy have the same characteristic classes. We refer the reader to [7] for proofs and further details.

6. K -theory

It is well-known that K -theory is the most efficient of all cohomology theories admitting a geometric description. Since we have a notion of representation of a Lie algebroid it is therefore natural to look at the possibility of constructing a K -theory in the context of Lie algebroids. A first attempt in constructing a K^0 -functor was made by Ginzburg in [16] and we recall his construction in this section. This should by no means be considered as the final word on this theory. In fact, we have good indications that a K -theory

based on the notion of representations up to homotopy would be more efficient, and this is the subject of current investigation.

We denote by $\text{Vect}_A(M)$ the semi-ring of equivalence classes of representations of a Lie algebroid $\pi : A \rightarrow M$. The following example shows that this semi-ring is usually too large.

Example 6.1. Consider the trivial Lie algebroid $\pi : A \rightarrow M$ of rank r , and let $p : E \rightarrow M$ be an ordinary vector bundle over M . Any family $\{\sigma_x\}_{x \in M}$ of representations of the abelian Lie algebras A_x on the vector space E_x ,

$$\sigma_x : A_x \rightarrow \text{End}(E_x),$$

determines a representation of A :

$$\nabla_{\alpha s}|_x = \sigma_x(\alpha(x)) \cdot s(x).$$

We must therefore somehow reduce the number of relevant representations. Ginzburg in [16] suggests reducing the number by identifying representations which can be deformed into one another.

Definition 6.2. Let E_0 and E_1 be representations of a Lie algebroid A . We say that E_0 and E_1 are *deformation equivalent* if they are isomorphic to representations that can be connected by a family E_t of representations of A .

This clearly defines an equivalence relation on $\text{Vect}_A(M)$ which respects the semi-ring structure. Denote then by $\text{Vect}_A^{\text{def}}(M)$ the semi-ring of equivalence classes of representations of a Lie algebroid A .

Example 6.3. For a trivial Lie algebroid $\pi : A \rightarrow M$, all representations are homotopic to one another so $\text{Vect}_A^{\text{def}}(M) = \text{Vect}(M)$. In fact, if $p : E \rightarrow M$ is some representation with an associated flat connection ∇ , then $\bar{\nabla}^t \equiv t\nabla$ defines a family of flat connections giving a deformation of (E, ∇) to the trivial representation.

Although this is not the only possible way of reducing equivalence classes of representations (see, e.g., [16] for the related concept of *homogeneous* representations), we restrict our attention to the deformation equivalent classes, for this already gives us the flavor of any such theory.

Definition 6.4. The *K-ring* of A is the Grothendieck ring $K(A)$ associated with the semi-ring Vect_A .

To check that these rings are reasonable objects let us look at some examples:

Example 6.5. In Example 6.1, we saw that for a trivial Lie algebroid A the *K-ring* $K(A)$ is equal to the ordinary *K-ring* of M , i.e., $K(A) = K(M)$.

Example 6.6. Let $A = TM$. Then a representation $E \rightarrow M$ of A is just a vector bundle together with an (ordinary) flat connection ∇ , or equivalently, a representation of $\pi_1(M)$. Two representations are deformation equivalent iff the corresponding representations of $\pi_1(M)$ lie in the same path-connected component of the space of representations of $\pi_1(M)$. Hence, we conclude that $K(TM) = \pi_0(\text{Rep}(\pi_1(M)))$.

Example 6.7. Let $A = \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathfrak{g}^*$ be the transformation Lie algebroid associated with the coadjoint action $\text{ad}^*: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}^*)$. Also, let $p: E \rightarrow \mathfrak{g}^*$ be some representation of A , with an associated flat connection ∇ .

There is a Lie algebra representation $\rho: \mathfrak{g} \rightarrow V$, where $V = p^{-1}(0)$, naturally associated with the representation. It can be defined as follows: if $y \in \mathfrak{g}$ and $v \in V$ choose sections $\alpha \in \Gamma(A)$ and $s \in \Gamma(E)$ such that $\alpha(0) = y$ and $s(0) = v$. Then

$$\rho(y) \cdot v \equiv \nabla_\alpha s(0),$$

and it is easy to check that this is independent of all choices.

The Lie algebra representation ρ defines a new representation $(\bar{E}, \bar{\nabla})$ of the Lie algebroid A : \bar{E} is the trivial vector bundle $\mathfrak{g}^* \times V \rightarrow \mathfrak{g}^*$, and $\bar{\nabla}$ is the unique flat connection which for a constant section $s(x) = v$ satisfies

$$\bar{\nabla}_\alpha s = \rho(\alpha) \cdot v$$

(here we identify $\alpha \in \Gamma(A)$ with a function $\alpha: \mathfrak{g}^* \rightarrow \mathfrak{g}$).

One can check (see [16]) that the representations (E, ∇) and $(\bar{E}, \bar{\nabla})$ are deformation equivalent. Moreover, two representations (E_0, ∇^0) and (E_1, ∇^1) are deformation equivalent iff the associated Lie algebra representations ρ_0 and ρ_1 are in the same path component of the space of representations of \mathfrak{g} . Hence, we conclude that $K(\mathfrak{g}^*) = R(\mathfrak{g})$, the ring of representations of \mathfrak{g} .

To complete the properties of this K -theory we consider *Morita equivalence* of Lie algebroids. The definition is based on the notion of *pull-back Lie algebroid* due to Higgins and Mackenzie (see [19]): we start with a Lie algebroid $A \rightarrow M$ and we consider a surjective submersion $\phi: Q \rightarrow M$. Then the pull-back Lie algebroid ϕ^*A completes the diagram

$$\begin{array}{ccc} \phi^*A & \xrightarrow{\hat{\phi}} & A \\ \downarrow & & \downarrow \\ Q & \xrightarrow{\phi} & M \end{array}$$

so that $\hat{\phi}: \phi^*A \rightarrow A$ is a morphism of Lie algebroids. As a vector bundle, ϕ^*A is given by:

$$\phi^*A = \{(\alpha, X) \in \phi^*A \times TQ: \# \alpha = d\rho \cdot X\},$$

where ϕ^*A is the usual pull-back of vector bundles. For the anchor one takes projection into the second factor, while the bracket is given by:

$$[(f\alpha, X), (g\beta, Y)] = (fg[\alpha, \beta] + X(g)\alpha - Y(f)\beta, [X, Y]),$$

whenever α, β are pull-backs of sections of A , $f, g \in C^\infty(Q)$ and $X, Y \in \mathcal{X}(Q)$. In the sequel we shall use ϕ^*A to denote the pull-back construction in the category of Lie algebroids and ϕ^*A to denote the pull-back construction in the category of vector bundles. Since we have $\phi^*A \simeq \phi^*A \oplus \text{Ker } \phi_*$, these constructions only coincide if ϕ is a covering.

Definition 6.8. Two Lie algebroids $A_1 \rightarrow M_1$ and $A_2 \rightarrow M_2$ are called *Morita equivalent* if there exists a pair of surjective submersions, with simply connected fibers,

$$\begin{array}{ccc} & Q & \\ \phi_1 \swarrow & & \searrow \phi_2 \\ M_1 & & M_2 \end{array}$$

such that the pull-back Lie algebroids $\phi_1^* A_1$ and $\phi_2^* A_2$ are isomorphic.

Remark 6.9. Morita equivalence was first introduced in the context of Poisson manifolds by Xu in [35], and further studied by Ginzburg and Lu in [18]. This definition makes sense only for *integrable* Poisson manifolds. For Lie algebroids, the definition above is due to Ginzburg [16], and is a linear version of the notion of Morita equivalence of Lie groupoids (see, e.g., [6]). For Poisson manifolds, this notion of Morita equivalence (applied to the cotangent bundle Lie algebroids) is weaker than Xu's definition of Morita equivalence, but it makes sense for *all* Poisson manifolds. For this reason it is called in [16] *weak Morita equivalence*. The advantage of this definition is that the invariants we have been discussing are in fact (weak) Morita invariants. The notion of weak Morita equivalence is just one of several possibilities (see the discussion in [16]).

Let E be a representation of the Lie algebroid A . Then $\phi^* E$ is naturally a representation of $\phi^* A$: the flat connections on E and $\phi^* E$ are related by

$$\tilde{\nabla}_{(\alpha, X)}(fs) = X(f)s + f\nabla_\alpha s,$$

whenever s is the pull-back of a section of E . Conversely, since the $\text{Ker } d\phi$ -action on $\phi^* E$ coincides with the action obtained from the natural flat connection on $\phi^* E$ along the ϕ -fibers, it follows that pull-back induces a bijection

$$\text{Vect}_A(M) \longleftrightarrow \text{Vect}_{\phi^* A}(Q).$$

Therefore, the definition implies that Morita equivalent Lie algebroids have the same representations:

Theorem 6.10 (Invariance under Morita Equivalence [6,16]). *Let $\pi_1: A_1 \rightarrow M_1$ and $\pi_2: A_2 \rightarrow M_2$ be Morita equivalent Lie algebroids. Then $\text{Vect}_{A_1}(M_1) \simeq \text{Vect}_{A_2}(M_2)$ and $\text{Vect}_{A_1}^{\text{def}}(M_1) \simeq \text{Vect}_{A_2}^{\text{def}}(M_2)$. In particular, we have:*

$$K(A_1) \simeq K(A_2).$$

One can also show, that Morita equivalent Lie algebroids have isomorphic zero and first Lie algebroid cohomology groups (see [16]). If one assumes further that the ϕ_i -fibers are n -connected, then one can show that they have isomorphic Lie algebroid cohomologies $H^k(A_1) \simeq H^k(A_2)$, for all $k \leq n$ (see [6, 16]).

Most of, if not all, the properties of Morita equivalence, can be reduced to the statement that Morita equivalent Lie algebroids have the same orbit space. In fact, the algebroids $\phi^* A_1$ and $\phi^* A_2$ obviously have isomorphic foliations and isomorphic transverse structures. Hence, the assignment $L \mapsto \phi_2(\phi_1^{-1}(L))$ gives a bijection between the leaves of A_1 and the leaves of A_2 . Moreover, if $x_1 = \phi(q)$ and $x_2 = \phi(q)$, it follows from the pull-back construction that the transverse Lie algebroid structures to A_1 at x_1 and to A_2 at x_2 , are isomorphic. Hence:

Theorem 6.11. *Let A_1 and A_2 be Morita equivalent Lie algebroids. Then there exists a 1 : 1 correspondence between the leaves of A_1 and the leaves of A_2 . Moreover, the transverse Lie algebroid structures of corresponding leaves are isomorphic.*

At this stage it is not clear how one can extend the results of this section to representations up to homotopy, an aspect of Lie algebroid theory we feel deserves further investigation.

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